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Structured perturbation of the Brunovsky form: A particular case ☆

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Abstract

We study the variation of the feedback invariants of a complex rectangular $n \times (n + m)$ matrix when we make small additive perturbations on the last m columns, in the particular case when the square $n \times n$ submatrix is similar to a block diagonal matrix, with two blocks in the diagonal associated with the controllable part and the noncontrollable part of the pair (A, B), respectively.

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1. Introduction

In the last decades the change of matrix canonical forms by means of small additive perturbations has been studied (see, for example, [4,9,11,12,14,16]). In these problems all the elements of the original matrices can be changed. In other cases only the elements in some predetermined positions can be perturbed, which is a kind of structured perturbation problem (see [5,6,7,8,10]). These last problems are also related to the completion problems, since a part of the matrix is fixed (see, for example, [2,3,18,20,21,22,24]).

In [5] rectangular matrices $[A\ B] \in \mathbb{C}^{n\times(n+m)}$ are considered and two different problems are studied. In the "necessary condition problem" we prove that there exists a neighbourhood of B such that for all the matrices B' in this neighbourhood the feedback invariants of (A, B') satisfy some necessary conditions.

In the "invariant prescription problem" we prove that these conditions are necessary and sufficient to find in every neighbourhood of B a matrix B' such that (A, B') has some prescribed numbers and polynomials as feedback invariants, in some particular cases. These conditions are not sufficient to solve the invariant prescription problem in the general case (see [5]).

In this paper, we give new necessary conditions for the case when the square matrix A is similar to a block diagonal matrix, with two blocks in the diagonal associated to the controllable and noncontrollable part of the pair (A, B), respectively.

These new conditions also turn out to be necessary and sufficient to solve the problem of prescription in the particular case when the two blocks in the diagonal of the matrix similar to A have disjoint spectra.

The paper is organized as follows: Section 2 is devoted to notation, definitions and previous results; in Section 3 we study the equivalence relation associated with this problem; in Section 4 we give the necessary condition theorem; in Section 5 we solve the prescription problem in the case when the blocks in the diagonal of A have disjoint spectra and in Section 6 we compare the necessary conditions obtained in [5] with those obtained in Section 4.

2. Notation, definitions and previous results

A *partition* is a finite or infinite sequence of nonnegative integers almost all zero,

$$a = (a_1, a_2, \ldots).$$

We denote by $\ell(a)$ the *length* of a, i.e., the number of the components different from zero.

The *conjugate partition* of a, $\overline{a} = (\overline{a}_1, \overline{a}_2, \ldots)$, is defined by

$$\overline{a}_k := \mathrm{Card}\{i : a_i \geqslant k\}.$$

We will use the symbol $\prec$ to mean *majorization* in the Hardy–Littlewood–Pólya sense (see [15]); i.e., if $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are two partitions and we denote by $a_{[1]} \geqslant \cdots \geqslant a_{[n]}$ and by $b_{[1]} \geqslant \cdots \geqslant b_{[n]}$ the components of a and b ordered in nonincreasing order, respectively, then

$$a \prec b \Leftrightarrow \begin{cases} \sum_{i=1}^{k} a_{[i]} \leqslant \sum_{i=1}^{k} b_{[i]}, & 1 \leqslant k \leqslant n-1, \\ \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i. \end{cases}$$

We will use the symbol $\prec\!\!\prec$ to mean *weak majorization*, i.e.

$$a \prec\!\!\prec b \Leftrightarrow \sum_{i=1}^{k} a_{[i]} \leqslant \sum_{i=1}^{k} b_{[i]}, \quad 1 \leqslant k \leqslant n.$$

We define $a \cup b$ as the partition whose components are those of a and b reordered in nonincreasing order.

We define $a + b$ to be the partition whose ith component is $(a + b)_i = a_{[i]} + b_{[i]}$.

The following properties are satisfied:

(1) $a \prec b \Leftrightarrow \overline{b} \prec \overline{a}$,
(2) $\overline{a \cup b} = \overline{a} + \overline{b}$.

We will denote by $\mathbb{F}$ an arbitrary field.

Let $X \in \mathbb{F}^{m\times n}$, with $m \leqslant n$. We will call *invariant factors* of X, the invariant factors of the polynomial matrix $[sI_m\ 0] - X$. We will denote by $d(\alpha)$ the degree of a polynomial α.

We will denote by $\Lambda(X) := \{\lambda_1, \dots, \lambda_v\}$ the spectrum of X, i.e., the set of the elements of the algebraic closure of the field $\mathbb{F}$ which are eigenvalues of the matrix X.

We will call *chain* a sequence of polynomials ordered by means of the divisibility order.

Let $\gamma_1|\cdots|\gamma_m$ and $\gamma_1'|\cdots|\gamma_m'$ be given monic polynomials. Let $\gamma = (\gamma_1, \dots, \gamma_m)$ and $\gamma' = (\gamma_1', \dots, \gamma_m')$ be the chains formed by these polynomials. We will say that γ' is *majorized* by γ and we will denote it by $\gamma' \prec \gamma$ if

$$\gamma_1' \cdots \gamma_i' | \gamma_1 \cdots \gamma_i, \quad \text{for } i = 1, \dots, m-1$$
$$\text{and} \quad \gamma_1' \cdots \gamma_m' = \gamma_1 \cdots \gamma_m.$$

We will say that γ' is *weakly majorized* by γ and we will denote it by $\gamma' \prec\!\!\prec \gamma$ if

$$\gamma_1' \cdots \gamma_i' | \gamma_1 \cdots \gamma_i, \quad \text{for } i = 1, \dots, m.$$

Remark 2.1. From now on, whenever the subindex of a polynomial of a chain is less than 1, we will consider that the polynomial is equal to 1.

We will call *companion matrix* of a monic polynomial $s^n - c_n s^{n-1} - \cdots - c_2 s - c_1 \in \mathbb{F}[s]$ a matrix of the form

$$\begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix}.$$

For a given matrix pair $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$, $\mathscr{C}(A, B)$ denotes, indifferently, the *controllability matrix* of (A, B), i.e., $[B\ AB\ \cdots\ A^{n-1}B]$ or the controllability subspace of (A, B), i.e., the subspace generated by the columns of the controllability matrix. This pair is said to be completely controllable if $\operatorname{rank}(\mathscr{C}(A, B)) = n$.

We will identify matrix pairs $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ with rectangular matrices $[A\ \ B] \in \mathbb{F}^{n\times(n+m)}$; in this way the invariant factors of (A, B) are those of the polynomial matrix $[sI_n - A\ \ -B]$. Moreover, the controllability indices of (A, B) are defined as in page 138 of [19] and will be denoted by $k_1 \geqslant \cdots \geqslant k_{r_1} > k_{r_1+1} = \cdots = k_m = 0$. An alternative criterion for controllability is that all the invariant factors of (A, B) be equal to 1.

We will denote by $(r_1, r_2, \ldots)$ the partition of the Brunovsky indices, which is the conjugate partition of the partition of the controllability indices. That partition can also be obtained by means of the following equalities:

$$\sum_{j=1}^{i} r_j = \operatorname{rank}[B\ AB\ \cdots\ A^{i-1}B], \quad i = 1, \ldots, n.$$

Two pairs (A_1, B_1) and (A_2, B_2) are said to be *feedback equivalent*, denoted by $(A_1, B_1) \overset{f.e.}{\sim} (A_2, B_2)$, if there exist nonsingular matrices $P \in \mathbb{F}^{n\times n}$ and $Q \in \mathbb{F}^{m\times m}$ and a matrix $R \in \mathbb{F}^{m\times n}$ such that

$$(A_1, B_1) = (P^{-1}A_2P + P^{-1}B_2R, P^{-1}B_2Q).$$

A complete system of invariants for the feedback equivalence is the one formed by the invariant factors and the controllability indices.

A canonical form for the feedback equivalence is given by the Brunovsky canonical form. It can be found in [21] among many other places.

Lemma 2.2. *Let* $[A \quad B] \in \mathbb{F}^{n\times(n+m)}$, $\operatorname{rank}(B) = r_1$, $\operatorname{rank}(\mathscr{C}(A, B)) = n_1$, $k_1 \geqslant \cdots \geqslant k_{r_1} > k_{r_1+1} = \cdots = k_m = 0$ *be the controllability indices of* $[A\ B]$ *and* $\alpha_1|\cdots|\alpha_n$ *be its invariant factors. Let us assume that* $\alpha_i = 1$, *for* $i = 1, \ldots, s$ *and* $d(\alpha_{s+1}) \geqslant 1$. *Then there exists a matrix* $[A_c\ B_c] \in \mathbb{F}^{n\times(n+m)}$ *feedback equivalent to* $[A\ B]$ *which satisfies the following conditions*:

(i) $A_c = \operatorname{diag}(M, N)$, $M \in \mathbb{F}^{n_1\times n_1}$ *and* $N \in \mathbb{F}^{(n-n_1)\times(n-n_1)}$,
(ii) $B_c = \begin{bmatrix} H \\ 0 \end{bmatrix}$, *where* $H = [\overline{H} \quad 0] \in \mathbb{F}^{n_1\times m}$ *and* $\overline{H} \in \mathbb{F}^{n_1\times r_1}$,
(iii) (M, H) *is a completely controllable pair and* $k_1, \ldots, k_m$ *are its controllability indices*,
(iv) $M = \operatorname{diag}(M_1, \ldots, M_{r_1})$, *where* M_i *is the companion matrix of* s^{k_i}, $i = 1, \ldots, r_1$,
(v) $\overline{H} = \begin{bmatrix} E_1 \\ \vdots \\ E_{r_1} \end{bmatrix}$, *where* $E_i = \begin{bmatrix} 0 \\ e_i \end{bmatrix} \in \mathbb{F}^{k_i\times r_1}$ *and* e_i *is the* i*th row of* I_{r_1},
(vi) $N = \operatorname{diag}(N_1, \ldots, N_{n-s})$, *where* N_i *is the companion matrix of the invariant factor* α_{s+i}, $i = 1, \ldots, n-s$.

We will denote by $(h_1, \ldots, h_m)$ the partition of the Hermite indices, defined as in [23]. Let us consider the following columns of the controllability matrix

$$\{b_1, Ab_1, \ldots, A^{n-1}b_1, b_2, Ab_2, \ldots, A^{n-1}b_2, \ldots, b_m, Ab_m, \ldots, A^{n-1}b_m\},$$

by choosing from left to right the first n_1 linearly independent columns we obtain the following basis of $\mathscr{C}(A, B)$,

$$\mathscr{B}_h = \{b_1, Ab_1, \ldots, A^{h_1-1}b_1, b_2, Ab_2, \ldots, A^{h_2-1}b_2, \ldots, b_m, Ab_m, \ldots, A^{h_m-1}b_m\},$$

where we agree that $h_i = 0$ if the column b_i has not been selected. Then the partition $h = (h_1, h_2, \ldots, h_m)$ is the partition of the *Hermite indices* of (A, B).

The following property, which relates the controllability indices to the Hermite indices of a pair of matrices, can be seen in [23].

Proposition 2.3. *Let $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ be a pair of matrices with $(k_1, \ldots, k_m)$ as controllability indices and $(h_1, \ldots, h_m)$ as Hermite indices. Then*

$$(k_1, \ldots, k_m) \prec (h_1, \ldots, h_m).$$

Remark 2.4. In the case when the pair (A, B) is in the Brunovsky canonical form we have that $(k_1, \ldots, k_m) = (h_1, \ldots, h_m)$.

Now we are going to state two previous results related to completion problems. In the first one the relationship between the invariants of a pair and those of the corresponding square matrix is given.

Theorem 2.5 [22]. *Let $A \in \mathbb{F}^{n\times n}$ and let $\omega_1|\cdots|\omega_n$ be its invariant factors. Let $\varphi_1|\cdots|\varphi_n$ be monic polynomials and let $k_1 \geqslant \cdots \geqslant k_{r_1}$ be given positive integers. Then there exists a matrix $B \in \mathbb{F}^{n\times m}$ with $\operatorname{rank}(B) = r_1$ such that $\varphi_1, \ldots, \varphi_n$ are the invariant factors and $k_1, \ldots, k_{r_1}$ are the nonnull controllability indices of $[A\ B]$ if and only if*

(i) $\omega_{i-r_1}|\varphi_i|\omega_i$, *for* $i = 1, \ldots, n$,
(ii) $(k_1, \ldots, k_{r_1}) \prec (d(\theta_{r_1}), \ldots, d(\theta_1))$,
where

$$\theta_j = \frac{\prod_{i=1}^{n+j} \operatorname{lcm}(\varphi_{i-j}, \omega_{i-r_1})}{\prod_{i=1}^{n+j-1} \operatorname{lcm}(\varphi_{i-j+1}, \omega_{i-r_1})}, \quad j = 1, \ldots, r_1.$$

In the second completion result we will take into account the remark which appears in page 194 of [24], in order to obtain some conditions which are more adapted to our problem.

Theorem 2.6 [24,17]. *Let $A_2 \in \mathbb{F}^{n_2\times n_2}$ be a matrix with $\alpha_1|\cdots|\alpha_{n_2}$ as invariant factors. Let $m, n_1, n = n_1 + n_2$, be positive integers. Let $h_1 \geqslant \cdots \geqslant h_m \geqslant 0$ and $l_1 \geqslant \cdots \geqslant l_m \geqslant 0$ be nonnegative integers. Let $\psi_1|\cdots|\psi_n$ be monic polynomials.*

Then there exist matrices $X \in \mathbb{F}^{n_2\times n_1}$ and $Y \in \mathbb{F}^{n_2\times m}$ and a controllable pair $(A_1, B_1) \in \mathbb{F}^{n_1\times n_1} \times \mathbb{F}^{n_1\times m}$ with $h_1, \ldots, h_m$ as Hermite indices such that

$$\left(\begin{bmatrix} A_1 & 0 \\ X & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ Y \end{bmatrix}\right)$$

has $\psi_1, \ldots, \psi_n$ as invariant factors and $l_1, \ldots, l_m$ as controllability indices if and only if:

$$\varphi_{i-r_1}|\psi_i|\varphi_i, \quad 1 \leqslant i \leqslant n, \tag{2.1}$$

$$(l_1, \ldots, l_m) \prec (h_1, \ldots, h_m) + (d(\delta_{r_1}), \ldots, d(\delta_1)), \tag{2.2}$$

where

$$\delta_j = \frac{\prod_{i=1}^{n+j} \operatorname{lcm}(\psi_{i-j}, \varphi_{i-r_1})}{\prod_{i=1}^{n+j-1} \operatorname{lcm}(\psi_{i-j+1}, \varphi_{i-r_1})}, \quad 1 \leqslant j \leqslant r_1,$$

with $r_1 = \ell(l)$, $\varphi_i := 1$, for $i \leqslant n_1$ and $\varphi_{i+n_1} := \alpha_i$, for $1 \leqslant i \leqslant n_2$.

If X is a complex matrix, we will denote by $\|X\|$ any submultiplicative matrix norm of X.

Now we enunciate some necessary conditions in the perturbation of a pair of matrices when only the columns of B are perturbed.

Theorem 2.7 [5]. *Let $A \in \mathbb{C}^{n\times n}$ and let $\omega_1|\cdots|\omega_n$ be its invariant factors. Let $[A\ B] \in \mathbb{C}^{n\times(n+m)}$ with $\varphi_1|\cdots|\varphi_n$ as invariant factors and $k_1 \geqslant \cdots \geqslant k_{r_1} > k_{r_1+1} = \cdots = k_m = 0$ as controllability indices.*

There exists $\varepsilon > 0$ such that if $\|[A\ B] - [A\ B']\| < \varepsilon$, $\varphi'_1|\cdots|\varphi'_n$ are the invariant factors and $k'_1 \geqslant \cdots \geqslant k'_{r'_1} > k'_{r'_1+1} = \cdots = k'_m = 0$ are the controllability indices of $[A\ B']$, then the following conditions hold:

(i) $\omega_{i-r'_1}|\varphi'_i|\omega_i$, *for $i = 1, \ldots, n$,*
(ii) $(k'_1, \ldots, k'_{r'_1}) \prec (d(\theta'_{r'_1}), \ldots, d(\theta'_1))$,
where

$$\theta'_j = \frac{\prod_{i=1}^{n+j} \operatorname{lcm}(\varphi'_{i-j}, \omega_{i-r'_1})}{\prod_{i=1}^{n+j-1} \operatorname{lcm}(\varphi'_{i-j+1}, \omega_{i-r'_1})}, \quad j = 1, \ldots, r'_1,$$

(iii) $\varphi' \lll \varphi$,
(iv) *if r and r' are the partitions of the Brunovsky indices of (A, B) and (A, B'), respectively, then*

$$r \lll r'.$$

Remark 2.8. If (A, B) is controllable, these necessary conditions are reduced to one: $k' \prec k$.

The conditions in Theorem 2.7 are also sufficient to solve the problem of prescription of invariants in the particular cases when the pair (A, B) is completely controllable or when it is completely incontrollable, i.e., when $B = 0$. They are also sufficient when B only has one column, (see [5]). We state the theorem when the pair is controllable because we will need it below.

Theorem 2.9 [5]. *Let $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ be a controllable pair with $k_1 \geqslant \cdots \geqslant k_{r_1} > k_{r_1+1} = \cdots = k_m = 0$ as controllability indices.*

Let $(k'_1, \ldots, k'_m)$ be a nonincreasing partition of integers.

For all $\varepsilon > 0$ there exists a matrix B' with $\|[A\ B] - [A\ B']\| < \varepsilon$, such that (A, B') is controllable and has $k'_1, \ldots, k'_m$ as controllability indices if and only if

$$(k'_1, \ldots, k'_m) \prec (k_1, \ldots, k_m).$$

In general, the conditions in Theorem 2.7 are not sufficient to find, as close to B as desired, a matrix B' in such a way that (A, B') has prescribed invariant factors and controllability indices, as it can be seen in the Example 7.4 and Section 8 of [5].

3. Equivalence relation associated with this problem

An equivalence relation associated with this problem in a natural way is the following one:

Let (A_1, B_1) and (A_2, B_2) be two pairs in $\mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$. We will say that (A_1, B_1) is PQ-*equivalent* to (A_2, B_2) and we will denote it by $(A_1, B_1) \overset{PQ}{\sim} (A_2, B_2)$ if there exist invertible matrices $P \in \mathbb{F}^{n\times n}$ and $Q \in \mathbb{F}^{m\times m}$ such that $(A_2, B_2) = (PA_1P^{-1}, PB_1Q)$.

The following lemma guarantees that if the problem is solved for a pair, it can be solved for another one in its class.

Lemma 3.1 [5]. *Let* $[A_1\ B_1] \in \mathbb{C}^{n\times(n+m)}$ *and let* $[A_2\ B_2] = [PA_1P^{-1}\ PB_1Q]$. *Then, in every neighbourhood of* $[A_1\ B_1]$ *there exists a matrix* $[A_1\ B_1 + E_1]$ *with* $k'_1 \geqslant \cdots \geqslant k'_m \geqslant 0$ *as controllability indices and* $\varphi'_1 | \cdots | \varphi'_n$ *as invariant factors if and only if in every neighbourhood of* $[A_2\ B_2]$ *there exists a matrix* $[A_2\ B_2 + E_2]$ *with these prescribed invariants.*

For the particular case when the matrix Q is the identity of order m we have the similarity of pairs, which will be denoted by $\overset{s}{\sim}$. We will use the Kalman decomposition (see [13], page 361), which we will denote by (A_K, B_K). Let $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ with $\varphi_1 | \cdots | \varphi_n$ as invariant factors and $k_1 \geqslant \cdots \geqslant k_m \geqslant 0$ as controllability indices. Then $(A, B) \overset{s}{\sim} (A_K, B_K)$ with

$$(A_K, B_K) = \left(\begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right) \in \mathbb{F}^{(n_1+n_2)\times(n_1+n_2)} \times \mathbb{F}^{(n_1+n_2)\times m}$$

where (A_1, B_1) is a controllable pair with $k_1 \geqslant \cdots \geqslant k_m \geqslant 0$ as controllability indices and A_2 is a square matrix with $\alpha_i = \varphi_{i+n_1}$, for $1 \leqslant i \leqslant n_2$, as invariant factors.

The following lemma shows the relationship between two Kalman decompositions of a pair.

Lemma 3.2 [25,1]. *Let* $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$. *Let*

$$\left(\begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right) \quad \textit{and} \quad \left(\begin{bmatrix} \overline{A}_1 & \overline{A}_3 \\ 0 & \overline{A}_2 \end{bmatrix}, \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix}\right)$$

be two different Kalman decompositions of the pair (A, B). *Then*

$$(A_1, B_1) \overset{s}{\sim} (\overline{A}_1, \overline{B}_1) \quad \textit{and} \quad A_2 \overset{s}{\sim} \overline{A}_2.$$

Therefore, for any given Kalman decomposition of the pair (A, B), the controllable part (A_1, B_1) and the square block A_2 are determined up to similarity.

Let $\left(\begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right)$ be a Kalman decomposition of (A, B) and let us suppose that $\Lambda(A_1) \cap \Lambda(A_2) = \emptyset$. Then there exists a matrix $L \in \mathbb{F}^{n_1\times n_2}$ such that $A_1L - LA_2 = A_3$ and it is easy to see that

$$\left(\begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right) \overset{s}{\sim} \left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right)$$

Thus, $\left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right)$ is also a Kalman decomposition of (A, B). We can denote such a block diagonal Kalman form by $K_D(A_1, A_2, B_1)$.

4. Necessary condition theorem

Now let us prove the following necessary condition theorem, in the case when the matrix A can be decomposed in diagonal blocks, associated with the controllable and noncontrollable parts of (A, B), respectively, non necessarily with disjoint spectra.

Theorem 4.1. *Let* $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ *be a pair similar to* $K_D(A_1, A_2, B_1)$ *with* $k_1 \geqslant \cdots \geqslant k_m \geqslant 0$ *as controllability indices and* $\varphi_1 | \cdots | \varphi_n$ *as invariant factors.*

There exists an $\varepsilon > 0$ *such that if* $\|[A\ B] - [A\ B']\| < \varepsilon$, $\varphi'_1, \ldots, \varphi'_n$ *are the invariant factors and* $k'_1, \ldots, k'_m$ *are the controllability indices of* (A, B'), *then*

(i) $\varphi_{i-r_1'}|\varphi_i'|\varphi_i, \quad 1 \leqslant i \leqslant n,$

(ii) $(k_1', \dots, k_m') \prec (k_1, \dots, k_m) + (d(\delta_{r_1'}'), \dots, d(\delta_1')),$

where

$$\delta_j' = \frac{\prod_{i=1}^{n+j} \operatorname{lcm}(\varphi_{i-j}', \varphi_{i-r_1'})}{\prod_{i=1}^{n+j-1} \operatorname{lcm}(\varphi_{i-j+1}', \varphi_{i-r_1'})}, \quad 1 \leqslant j \leqslant r_1',$$

with $r_1' = \ell(k')$.

Proof. With no loss of generality we can assume that $(A, B) = K_D(A_1, A_2, B_1)$.

The pair (A_1, B_1) is controllable. By Theorem 2.7 and Remark 2.8 there exists ε_1 such that for any matrix B_{11}' with $\|B_{11}' - B_1\| < \varepsilon_1$ we have that the pair (A_1, B_{11}') has controllability indices which are majorized by those of the pair (A_1, B_1).

Let $\varepsilon := \varepsilon_1$ and let B' be such that $\|B' - B\| < \varepsilon$.

Let $\varphi_1', \dots, \varphi_n'$ be the invariant factors and let $k_1', \dots, k_m'$ be the controllability indices of (A, B').

Let us consider $B' =: \begin{bmatrix} B_1' \\ Y \end{bmatrix}$, with $B_1' \in \mathbb{C}^{n_1 \times m}$ and let us call $(c_1, \dots, c_m)$ the controllability indices of (A_1, B_1').

We have that $\|B_1' - B_1\| < \varepsilon = \varepsilon_1$ and, by what have been said above, it holds that

$$(c_1, \dots, c_m) \prec (k_1, \dots, k_m). \tag{4.3}$$

Let (A_c, B_c) be the Brunovsky canonical form of (A_1, B_1').

Then

$$(A, B') = \left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1' \\ Y \end{bmatrix} \right) \overset{f.e.}{\sim} \left(\begin{bmatrix} A_c & 0 \\ X & A_2 \end{bmatrix}, \begin{bmatrix} B_c \\ Y' \end{bmatrix} \right),$$

where $X \in \mathbb{C}^{n_2 \times n_1}$ and $Y' \in \mathbb{C}^{n_2 \times m}$.

The Hermite indices of (A_c, B_c) are $c_1 \geqslant \cdots \geqslant c_m$, by Remark 2.4. By Theorem 2.6, we obtain (i) and

$$(k_1', \dots, k_m') \prec (c_1, \dots, c_m) + (d(\delta_{r_1'}'), \dots, d(\delta_1')).$$

From this and (4.3), (ii) can be deduced. □

5. Prescription theorem for pairs similar to $K_D(A_1, A_2, B_1)$ with $\Lambda(A_1) \cap \Lambda(A_2) = \emptyset$

Lemma 5.1. *Let* $(A_1, B_1) \in \mathbb{F}^{n_1 \times n_1} \times \mathbb{F}^{n_1 \times m}$ *and* $A_2 \in \mathbb{F}^{n_2 \times n_2}$ *such that* $\Lambda(A_1) \cap \Lambda(A_2) = \emptyset$.

Then, for any $X_1 \in \mathbb{F}^{n_2 \times n_1}$ *and* $Y_1 \in \mathbb{F}^{n_2 \times m}$ *there exists* $Y \in \mathbb{F}^{n_2 \times m}$ *such that*

$$\left(\begin{bmatrix} A_1 & 0 \\ X_1 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ Y_1 \end{bmatrix} \right) \overset{s}{\sim} \left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ Y \end{bmatrix} \right).$$

Proof. Let $X_1 \in \mathbb{F}^{n_2 \times n_1}$ and $Y_1 \in \mathbb{F}^{n_2 \times m}$.

Since $\Lambda(A_1) \cap \Lambda(A_2) = \emptyset$, there exists $L \in \mathbb{F}^{n_2 \times n_1}$ such that $A_2 L - L A_1 = X_1$. Then, if we define $Y := Y_1 + L B_1$, it is easy to see that the pairs are similar. □

Now we are going to prove the following invariant prescription theorem for pairs with a block diagonal Kalman form, where the blocks in the diagonal have disjoint spectra.

Theorem 5.2. *Let $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ be a pair similar to $K_D(A_1, A_2, B_1)$ with $k_1 \geqslant \cdots \geqslant k_m \geqslant 0$ as controllability indices and $\varphi_1 | \cdots | \varphi_n$ as invariant factors. Let us suppose that $\Lambda(A_1) \cap \Lambda(A_2) = \emptyset$.*

Let $k'_1 \geqslant \cdots \geqslant k'_{r'_1} > k'_{r'_1+1} = \cdots = k'_m \geqslant 0$ be integers and $\varphi'_1 | \cdots | \varphi'_n$ be monic polynomials.

For all $\varepsilon > 0$ there exists a matrix B' with $\|[A\ B] - [A\ B']\| < \varepsilon$, such that $\varphi'_1, \ldots, \varphi'_n$ are the invariant factors and $k'_1, \ldots, k'_m$ are the controllability indices of (A, B') if and only if

(i) $\varphi_{i-r'_1} | \varphi'_i | \varphi_i, \quad 1 \leqslant i \leqslant n,$

(ii) $(k'_1, \ldots, k'_m) \prec (k_1, \ldots, k_m) + (d(\delta'_{r'_1}), \ldots, d(\delta'_1)),$

where

$$\delta'_j = \frac{\prod_{i=1}^{n+j} \operatorname{lcm}(\varphi'_{i-j}, \varphi_{i-r'_1})}{\prod_{i=1}^{n+j-1} \operatorname{lcm}(\varphi'_{i-j+1}, \varphi_{i-r'_1})}, \quad 1 \leqslant j \leqslant r'_1.$$

Proof. There is no loss of generality if we suppose that $(A, B) = K_D(A_1, A_2, B_1)$.

Let us see that the conditions are necessary.

Let us suppose that for all $\varepsilon > 0$ there exists a matrix B' with $\|[A\ B] - [A\ B']\| < \varepsilon$, such that $[A\ B']$ has as invariant factors the prescribed $\varphi'_1, \ldots, \varphi'_n$ and as controllability indices the prescribed $k'_1, \ldots, k'_m$. If we consider ε to be the real number which appears in Theorem 4.1, then it is easy to see that the prescribed feedback invariants satisfy (i) and (ii).

Let us also prove that they are sufficient.

By Theorem 2.6 there exist matrices $X \in \mathbb{C}^{n_2\times n_1}$, $Y \in \mathbb{C}^{n_2\times m}$ and a controllable pair $(\widetilde{A}_1, \widetilde{B}_1) \in \mathbb{C}^{n_1\times n_1} \times \mathbb{C}^{n_1\times m}$ with $k_1, \ldots, k_m$ as Hermite indices such that

$$\left(\begin{bmatrix} \widetilde{A}_1 & 0 \\ X & A_2 \end{bmatrix}, \begin{bmatrix} \widetilde{B}_1 \\ Y \end{bmatrix}\right)$$

has $\varphi'_1, \ldots, \varphi'_n$ as invariant factors and $k'_1, \ldots, k'_m$ as controllability indices. Let $(c_1, \ldots, c_m)$ be the controllability indices of $(\widetilde{A}_1, \widetilde{B}_1)$. By Proposition 2.3,

$$(c_1, \ldots, c_m) \prec (k_1, \ldots, k_m). \tag{5.4}$$

Let $\varepsilon > 0$ and let $0 < \varepsilon_1 < \varepsilon$.

The controllability indices of (A_1, B_1) are $(k_1, \ldots, k_m)$. From (5.4), by Theorem 2.9 there exists a matrix B'_1 with $\|[A_1\ B_1] - [A_1\ B'_1]\| < \varepsilon_1$, such that (A_1, B'_1) is controllable and has $c_1, \ldots, c_m$ as controllability indices.

Since $(\widetilde{A}_1, \widetilde{B}_1)$ and (A_1, B'_1) are controllable with the same controllability indices,

$$(\widetilde{A}_1, \widetilde{B}_1) \overset{f.e.}{\sim} (A_1, B'_1).$$

But then,

$$\left(\begin{bmatrix} \widetilde{A}_1 & 0 \\ X & A_2 \end{bmatrix}, \begin{bmatrix} \widetilde{B}_1 \\ Y \end{bmatrix}\right) \overset{f.e.}{\sim} \left(\begin{bmatrix} A_1 & 0 \\ X' & A_2 \end{bmatrix}, \begin{bmatrix} B'_1 \\ Y' \end{bmatrix}\right)$$

with $X' \in \mathbb{C}^{n_2\times n_1}$ and $Y' \in \mathbb{C}^{n_2\times m}$.

As $\Lambda(A_1) \cap \Lambda(A_2) = \emptyset$, by Lemma 5.1 we have that,

$$\left(\begin{bmatrix} A_1 & 0 \\ X' & A_2 \end{bmatrix}, \begin{bmatrix} B'_1 \\ Y' \end{bmatrix}\right) \overset{s}{\sim} \left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B'_1 \\ Y'' \end{bmatrix}\right).$$

with $Y'' \in \mathbb{C}^{n_2\times m}$.

If $Y^{''}=0$ we have finished. In other case, let $\varepsilon_2>0$ be such that $\varepsilon_1+\varepsilon_2<\varepsilon$.

If we define $Z:=\frac{\varepsilon_2}{\|Y^{''}\|}Y^{''}$ we have that

$$(A,B')=\left(\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix},\begin{bmatrix}B_1'\\ Z\end{bmatrix}\right),$$

satisfies that $\|[A\ B]-[A\ B']\|<\varepsilon$, has $\varphi_1',\dots,\varphi_n'$ as invariant factors and $k_1',\dots,k_m'$ as controllability indices. □

6. Relationship between the two groups of conditions

Now let us see that when $\Lambda(A_1)\cap\Lambda(A_2)=\emptyset$, the conditions in Theorem 4.1 imply those in Theorem 2.7.

Theorem 6.1. *Let $\omega_1|\cdots|\omega_n$, $\varphi_1|\cdots|\varphi_n$ and $\varphi_1'|\cdots|\varphi_n'$ be the invariant factors of $A=\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}$, (A,B) and (A,B'), respectively. Let $k_1\geqslant\cdots\geqslant k_{r_1}>k_{r_1+1}=\cdots=k_m=0$ and $k_1'\geqslant\cdots\geqslant k_{r_1'}'>k_{r_1'+1}'=\cdots=k_m'=0$ be the controllability indices of (A,B) and (A,B'), respectively.*

Let us suppose that $\Lambda(A_1)\cap\Lambda(A_2)=\emptyset$.

The conditions:

(Ai) $\varphi_{i-r_1'}|\varphi_i'|\varphi_i,\quad 1\leqslant i\leqslant n$,

(Aii) $(k_1',\dots,k_m')\prec(k_1,\dots,k_m)+(d(\delta_{r_1'}'),\dots,d(\delta_1'))$,

where

$$\delta_j'=\frac{\prod_{i=1}^{n+j}\operatorname{lcm}(\varphi_{i-j}',\varphi_{i-r_1'})}{\prod_{i=1}^{n+j-1}\operatorname{lcm}(\varphi_{i-j+1}',\varphi_{i-r_1'})},\quad 1\leqslant j\leqslant r_1',$$

imply the conditions:

(Bi) $\omega_{i-r_1'}|\varphi_i'|\omega_i,\quad 1\leqslant i\leqslant n$,

(Bii) $(k_1',\dots,k_{r_1'}')\prec(d(\theta_{r_1'}'),\dots,d(\theta_1'))$,

where

$$\theta_j'=\frac{\prod_{i=1}^{n+j}\operatorname{lcm}(\varphi_{i-j}',\omega_{i-r_1'})}{\prod_{i=1}^{n+j-1}\operatorname{lcm}(\varphi_{i-j+1}',\omega_{i-r_1'})},\quad 1\leqslant j\leqslant r_1',$$

(Biii) $\varphi'\lll\varphi$,

(Biv) *if r and r' are the partitions of the Brunovsky indices of (A,B) and (A,B'), respectively, then*

$$r\lll r'.$$

Proof. Let us suppose that (Ai) and (Aii) are satisfied.

From (Ai), (Biii) can immediately be deduced.

From (Aii), we have that

$$r\cup\overline{(d(\delta_{r_1'}'),\dots,d(\delta_1'))}\prec r',$$

and (Biv) can be obtained.

By Theorem 2.5, we have that

$$\omega_{i-r_1}|\varphi_i|\omega_i, \quad 1 \leqslant i \leqslant n,$$

and thus, from (Ai), we obtain

$$\varphi'_i|\omega_i, \quad 1 \leqslant i \leqslant n.$$

Let $\beta_1|\cdots|\beta_{n_1}$ be the invariant factors of A_1.

By condition (i) of Theorem 2.5, applied to the controllable pair (A_1, B_1), we deduce that $\beta_i = 1$, for $1 \leqslant i \leqslant n_1 - r_1$. Let

$$\tau_i := 1, \quad 1 \leqslant i \leqslant n_2; \qquad \tau_{i+n_2} := \beta_i, \quad 1 \leqslant i \leqslant n_1.$$

Then, as $\Lambda(A_1) \cap \Lambda(A_2) = \emptyset$,

$$\omega_i = \varphi_i \tau_i, \quad 1 \leqslant i \leqslant n.$$

Since $r'_1 \geqslant r_1$,

$$\tau_{i-r'_1}|\tau_{i-r_1} = \beta_{i-n_2-r_1} = 1, \quad 1 \leqslant i \leqslant n.$$

Therefore, from (Ai),

$$\omega_{i-r'_1} = \tau_{i-r'_1}\varphi_{i-r'_1} = \varphi_{i-r'_1}|\varphi'_i, \quad 1 \leqslant i \leqslant n.$$

Now, we only have to prove (Bii).

By condition (ii) of Theorem 2.5, applied to the controllable pair (A_1, B_1), we have that

$$(k_1, \ldots, k_m) \prec (d(\beta_{n_1}), \ldots, d(\beta_1)).$$

That is, taking into account that $\tau_i = 1$ for $i = 1, \ldots, n - r'_1$, we obtain

$$(k_1, \ldots, k_m) \prec (d(\tau_n), \ldots, d(\tau_{n-r'_1+1})).$$

As a consequence from (Aii),

$$(k'_1, \ldots, k'_m) \prec (d(\tau_n), \ldots, d(\tau_{n-r'_1+1})) + (d(\delta'_{r'_1}), \ldots, d(\delta'_1)). \tag{6.5}$$

Let us see that

$$\theta'_1 \cdots \theta'_j|\tau_{n-r'_1+1} \cdots \tau_{n-r'_1+j}\delta'_1 \cdots \delta'_j, \quad 1 \leqslant j \leqslant r'_1 - 1, \tag{6.6}$$

$$\theta'_1 \cdots \theta'_{r'_1} = \tau_{n-r'_1+1} \cdots \tau_n \delta'_1 \cdots \delta'_{r'_1}. \tag{6.7}$$

For $1 \leqslant j \leqslant r'_1$,

$$\prod_{i=1}^{n+j} \operatorname{lcm}(\varphi'_{i-j}, \omega_{i-r'_1}) = \prod_{i=1}^{n+j} \operatorname{lcm}(\varphi'_{i-j}, \tau_{i-r'_1}\varphi_{i-r'_1}) \Bigg| \prod_{i=1}^{n+j} \tau_{i-r'_1} \operatorname{lcm}(\varphi'_{i-j}, \varphi_{i-r'_1}).$$

Then

$$\frac{\prod_{i=1}^{n+j} \operatorname{lcm}(\varphi'_{i-j}, \omega_{i-r'_1})}{\prod_{i=1}^{n} \varphi'_i} \Bigg| \frac{\prod_{i=1}^{n+j} \tau_{i-r'_1} \operatorname{lcm}(\varphi'_{i-j}, \varphi_{i-r'_1})}{\prod_{i=1}^{n} \varphi'_i}$$

$$= \prod_{i=n+1}^{n+j} \tau_{i-r'_1} \frac{\prod_{i=1}^{n+j} \operatorname{lcm}(\varphi'_{i-j}, \varphi_{i-r'_1})}{\prod_{i=1}^{n} \varphi'_i},$$

and (6.6) can be deduced. Moreover

$$\theta'_1 \cdots \theta'_{r'_1} = \frac{\prod_{i=1}^n \omega_i}{\prod_{i=1}^n \varphi'_i} = \frac{\prod_{i=1}^n \tau_i \varphi_i}{\prod_{i=1}^n \varphi'_i} = \tau_{n-r'_1+1} \cdots \tau_n \delta'_1 \cdots \delta'_{r'_1},$$

which is the equality (6.7).

Conditions (6.6) and (6.7) imply

$$(d(\tau_n), \ldots, d(\tau_{n-r'_1+1})) + (d(\delta'_{r'_1}), \ldots, d(\delta'_1)) \prec (d(\theta'_{r'_1}), \ldots, d(\theta'_1)),$$

and together with (6.5) this implies (Bii). □

Conditions (i) and (ii) of Theorem 4.1 are not sufficient to find, as close to B as we want, a matrix B' in such a way that the pair (A, B') has prescribed invariant factors and controllability indices, in the case when the spectra of A_1 and A_2 are not disjoint, as it can be seen in the Example 7.4 of [5], where these conditions are satisfied.

Conditions (i) and (ii) of Theorem 4.1 are not necessary, in the case when the matrix A can not be diagonalized in blocks A_1 and A_2, as it can be seen in the following example.

Counterexample

Let

$$[A\ B] = \left[\begin{array}{c||cccc||cc} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

The invariant factors of (A, B) are $\varphi = (1, 1, s, s, s(s-1))$, its controllability indices are $k = (1)$ and, as a consequence, those of Brunovsky are $r = (1)$.

Now let

$$[A\ B'] = \left[\begin{array}{c||cccc||cc} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right],$$

where ε is an arbitrarily small number. The invariant factors of (A, B') are $\varphi' = (1, 1, 1, 1, s^2(s-1))$, the controllability indices of (A, B') are $k' = (1, 1)$ and those of Brunovsky are $r' = (2)$.

Condition (i) does not hold because $s^2(s-1) = \varphi'_5 \nmid \varphi_5 = s(s-1)$.

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